Three-dimensional negative eddy viscosity effect on the onset of instability in some planar flows

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Three-dimensional linear stability of the rectangular cell flow $\Psi = \cos kx \cos y$ $(0<k<1)$ to long-wave disturbances is investigated numerically. Owing to the spatial periodicity of the flow the disturbances take the form $\exp[\sigma t + i(\alpha kx + \beta y + \gamma z)]F(kx,y)$ $(0<\alpha<0.5, 0<\beta<0.5$, and $0<\gamma<\infty)$, where F has the same periodicity as the main flow. It is found that the *critical* Reynolds number is determined by the three-dimensional *large-scale* modes in the range $0.55 \lesssim k \lesssim 0.71$. The direction of the critical mode $(\alpha=0, \beta\approx\gamma)$ is independent of the anisotropic parameter k.

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Stability of flows periodic in space and, possibly in time, is currently a topic of active interest in the Navier-Stokes flows. Cellular flows that are constructed by a combination of sine and cosine functions may be regarded as a simple model of the secondary flows with the two-dimensional spatial structures [1]. One advantage to treat this type of flow as a substitute for more realistic flows is to reduce the burden of the numerical and analytical task. We hope that this model shares some common characteristics with more general two-dimensional flows which are frequently observed in laboratory experiments and natural phenomena. In addition to this purpose, several researchers have treated them in order to discuss the possibility of the inverse energy cascade and spontaneous organization of large-scale structures in hydrodynamic turbulence [2].

Instability of a simple parallel flow, i.e., the Kolmogorov flow ($\Psi=\cos y$), was first investigated and it was shown that the flow becomes unstable to the *large-scale* modes, whose structure is almost uniform. Its growth rate is proportional to the square of the wave numbers; this type of the instability is often called the negative eddy viscosity effect [3]. As an extension of the parallel periodic flows, stability of some two-dimensional periodic flows such as a rhombic cell flow and a triangular cell flow are also treated [1,2]. Some anisotropic systems exhibit a negative eddy viscosity effect which is similar to the case of the Kolmogorov flow, that is, a two-dimensional large-scale secondary flow appears at a supercritical Reynolds number. Previous studies on the stability of the two-dimensional flows are restricted to treating the two-dimensional disturbances. A search for the appearance of *three*-dimensional large-scale structures in purely two-dimensional flows $[U(x,y), V(x,y), 0]$ has not been carried out yet, except for our recent works [4] (to the authors' knowledge).

In the linear stability theory of the two-dimensional *parallel* flow, Squire's theorem guarantees that the primary instability occurs in the development of the two-dimensional disturbances [5]. However, this theorem is not extended in general to the problem of the nonparallel flow; a three-dimensional disturbance may play a decisive role in determining the critical Reynolds number. Based on this motivation, the three-dimensional stability of some spatially periodic flows has been investigated numerically. In rhombic cell flows [4], the three-dimensional effects on the critical modes always reduce the instability, which leads us to conclude that the critical Reynolds number is determined by the two-dimensional disturbances in this case. However, we have no proof that the critical Reynolds number is determined by a two-dimensional disturbance even if we limit our concern to long-wave disturbances.

In this paper, by solving the eigenvalue problem, we present what we believe to be the first example that a three-dimensional disturbance determines the critical Reynolds number of the flows governed by the Navier-Stokes equation. We consider a simple rectangular cell flow represented by the stream function:

$$\Psi(x,y) = \cos kx \cos y \ , \tag{1}$$

where the parameter k $(0<k<1)$ represents the anisotropy of the flow. It is also expressed by $[\cos(kx+y)+\cos(kx-y)]/2$. An example of the case $k=0.6$ is shown in Fig. 1. Using it as a counterexample we can prove explicitly that Squire's theorem cannot be extended to general two-dimensional flows. Another purpose of this work is to investigate whether a three-dimensional large-scale flow can appear spontaneously under a small-scale two-dimensional forcing or in the two-dimensional flows which are uniform in the vertical direction. It should be noted that Bayly [6] investigated a three-dimensional instability of this flow by solving the time-evolution of the linearized Navier-Stokes equation. In his work his attention is focused on the short-wave instability in (almost) inviscid flows related to the elliptical instability [7]. The large-scale instability cannot be treated by this method.

Below we formulate our problems following our previous works [4]. We consider an incompressible flow governed by the Navier-Stokes equation, appropriately nondimensionalized, with the forcing term:

$$\partial_t u_i + (u_j \partial_j) u_i + \partial_i p = R^{-1} \partial_j^2 u_i + f_i \ , \tag{2}$$

$$\partial_j u_j = 0 \ , \tag{3}$$

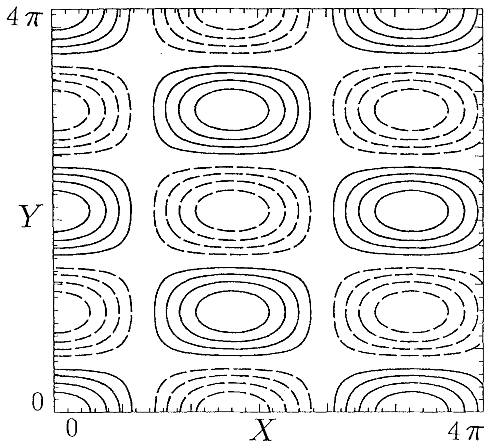

FIG. 1. An example of the streamlines of the main flow: $\Psi = \cos kx \cos y$ with $k = 0.6$. The bold line represents counterclockwise rotation while the dashed line represents clockwise rotation.

where u_i is the velocity; p is the pressure; f_i is the external force arranged to provide a steady two-dimensional flow $\mathbf{u}_0 = (V_1, V_2, 0)$, where $V_1 = \partial_2 \Psi$ and $V_2 = -\partial_1 \Psi$ in terms of the stream function Ψ; $\partial_t = \partial/\partial t$; $\partial_k = \partial/\partial x_k$; R is the Reynolds number; and we use Einstein's convention for summation.

Applying twice the curl in succession to Eq. (2) and using Eq. (3), we have

$$(\partial_t - R^{-1}\Delta)\Delta v_i + \Delta(v_k \partial_k v_i) - \partial_i(\partial_j v_k \partial_k v_j) - \partial_i(\partial_k f_k) + \Delta f_i = 0 \ . \quad (4)$$

Here we consider the instability of $\mathbf{u}_0$ to the three-dimensional disturbances $\vec{u}' = (v_1, v_2, v_3)$ and p'. It should be stressed that they depend on all spatial coordinates (x_1, x_2, x_3). By substituting $\vec{u} = \vec{u}_0 + \vec{u}'$ into Eq. (4) and neglecting the quadratic terms of disturbance, we have a system of equations as follows:

$$(\partial_t - R^{-1}\Delta)\Delta v_i + \Delta(V_1 \partial_1 v_i + V_2 \partial_2 v_i + v_1 \partial_1 V_i + v_2 \partial_2 V_i) - 2\partial_i(\partial_1 V_1 \partial_1 v_1 + \partial_1 V_2 \partial_2 v_1 + \partial_2 V_1 \partial_1 v_2 + \partial_2 V_2 \partial_2 v_2) = 0 \ , \quad (5)$$

$$\partial_3 v_3 = -(\partial_1 v_1 + \partial_2 v_2) \ , \quad (6)$$

where $i = 1, 2$. The disturbance must be subject to the boundary condition: $v_i(\pm\infty, y, t) = v_i(x, \pm\infty, t) < \infty$. It should be noticed here that in Eq. (5) the first two equations governing v_1 and v_2 are separated from the third which determines v_3 from v_1 and v_2.

The main flow (1) satisfies the equation $\Delta\Psi = -(k^2+1)\Psi$; it belongs to the exact solutions of the Euler equation. The magnitude of both wave numbers of respective terms in Eq. (1) is equal to $\sqrt{k^2+1}$. Since this main flow has the periodic structure $\Psi(x + 2\pi/k, y) = \Psi(x, y + 2\pi) = \Psi(x, y)$, the solution of Eq. (5) can be found by the Floquet theory system in the form

$$v_i = \exp[\sigma t + i(\alpha k x + \beta y + \gamma z)] F_i(kx, y) \ , \quad (7)$$

where σ is a complex growth rate, α and β are the Floquet exponents, being real in accordance with the boundary condition, and γ is the wave number perpendicular to the main flow. The function $F_i(kx, y)$ must be subject to the periodic condition: $F_i(kx + 2\pi, y) = F_i(kx, y + 2\pi) = F_i(kx, y)$. The function $F_i(kx, y)$ specified by Eq. (7) is given by

$$[F_1(kx, y), F_2(kx, y)] = \sum [a_{mn}, b_{mn}] \exp[i(mkx + ny)] \ , \quad (8)$$

where the summation takes all combinations of the integers m and n. Substitution of Eq. (8) into Eq. (5) gives an infinite set of algebraic eigenvalue equations of a_{mn}'s and b_{mn}'s, whose forms are omitted for the sake of space. The eigenvalue equations depend on five free parameters: α, β, γ, k, and R. The variation ranges of α and β are reduced to $[0, 1/2]$ without loss of generality, but that of γ is $[0, \infty]$.

To find the critical Reynolds number R_c we set $\sigma = 0$ on the assumption of the principal of exchange of stability and solve the eigenvalue problem for the Reynolds number R instead of solving the eigenvalue problem for the growth rate σ. After obtaining the value of R_c we confirm that the sign of the real part of σ actually changes as the Reynolds number R increases through this value. The numerical calculation is performed for a cutoff $M \times M$ matrix. The cutoff number M is defined by $2(2N_x + 1)(2N_y + 1)$, where N_x and N_y are truncation numbers in the x and the y directions, respectively. We solve the eigenvalue problem numerically using the modified QR method with double precision. The numerical calculations have been carried out using a 242×242 matrix with $N_x = N_y = 5$ in most cases. We shall use the values whose difference between the cases with $N_x = N_y = 4$ and $N_x = N_y = 5$ is less than 0.1% of them.

Of our concern is the behavior of the growth rate in the vicinity of the origin ($\alpha = \beta = \gamma = 0$). In order to clarify the dependence of the origin on the direction of the limits of these parameters to the origin, we use the spherical polar coordinate

$$(\alpha, \beta, \gamma) = \epsilon(\cos\phi \cos\theta, \cos\phi \sin\theta, \sin\phi) \ , \quad (9)$$

where $\epsilon = \sqrt{\alpha^2 + \beta^2 + \gamma^2}$, $\theta = \arctan(\beta/\alpha)$, and $\phi = \arctan(\gamma/\sqrt{\alpha^2 + \beta^2})$. Typically we select $\epsilon = 10^{-3}$ or 10^{-4}. It should be noted that we have not used the conventional spherical coordinates. The angle ϕ measures the three-dimensionality of the disturbances (i.e., $\phi = 0$ corresponds to the purely two-dimensional disturbance).

Before we describe the present results of the three-dimensional effects on the stability, we sum up the results of the critical Reynolds number to the two-dimensional disturbances [8]. The results of the critical Reynolds numbers for various values of the anisotropic parameter k between 0 and 1 are shown in Fig. 2. Two types of modes are indicated in Fig. 2. The large-scale mode is denoted by the dashed curve, whose structure is uniform in space. The critical modes are given along the direction $\theta \approx \arctan k$. The growth rate is proportional to the square of the exponents [i.e., $\sigma = \sigma_2 \epsilon^2 + \cdots$, $\sigma_2 = \sigma_2(\theta, R)$ and $\partial\sigma_2/\partial\theta \neq 0$]. The *periodic* mode is

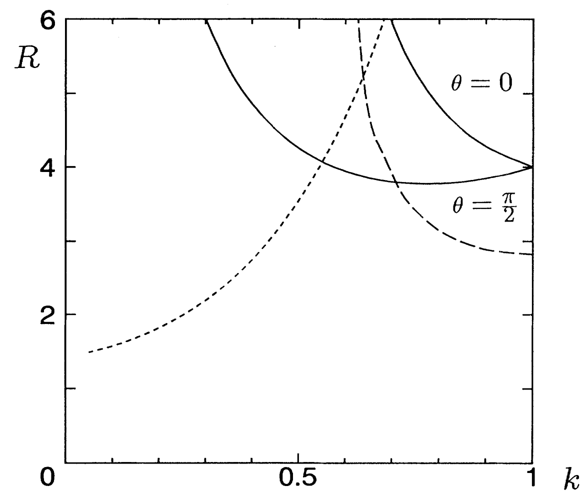

FIG. 2. The critical Reynolds number against the anisotropic parameter k. The lower solid line represents the 3D minimum Reynolds number: $R(k,\theta=\pi/2)=\min_\phi R(k,\theta=\pi/2,\phi)$. The angle ϕ is close to $\pi/4$ as shown in Table I. The upper solid line represents the 3D minimum Reynolds number: $R(k,\theta=0)=\min_\phi R(k,\theta=0,\phi)$. The angle ϕ is close to $\pi/4$. Dashed line: 2D critical Reynolds number of the large-scale mode; dotted line: 2D critical Reynolds number of the periodic mode.

denoted by the dotted curve, whose structure has the same basic periodicity [i.e., $F(kx,y)\neq 1$.] The growth rate [i.e., $\sigma=\sigma_0+\sigma_2\epsilon^2+\cdots$, $\sigma_2=\sigma_2(\theta,R)<0$, $\partial\sigma_0/\partial\theta=0$] is independent of the direction θ in parameter space.

In our previous study [4] we investigated the three-dimensional stability of the square cell flow which corresponds to the case with $k=1$ here, and obtained the result that the growth rate of the critical modes with specified θ decreases owing to the three-dimensional effect. For some θ the three-dimensional effect reduces the marginal Reynolds number. All modes are *large-scale* modes. In the range $0<\theta<0.625$ the Reynolds number seems to become infinite to two-dimensional disturbances and takes the minimum at $\phi(\approx\pi/4)$, but this is larger than the value of two-dimensional disturbance $(\theta,\phi)=(\pi/4,0)$. Therefore we expect that the three-dimensional disturbance may determine the critical value of R if the two-dimensional disturbances in *any* direction show the positive eddy viscosity (i.e., $R\to\infty$ for any θ with $\phi=0$).

Along the line of this idea we examine the case with $k=0.6$. Note that the dashed line diverges at $k=1/\sqrt{3}$ where this mode seems to have positive eddy viscosity in any direction [8]. The variations of Reynolds number against ϕ for several values of θ are shown in Fig. 3. The straight line represents the critical Reynolds number R_2 of the periodic mode by two-dimensional disturbances which is independent of θ and ϕ. The revised critical Reynolds number is found at the critical direction: $(\theta_c,\phi_c)=(\pi/2,0.815)$. If we choose the value closer to $k=1/\sqrt{3}$, then R_2 becomes larger. We did not attempt to determine R_2 around $k=1/\sqrt{3}$ here because it requires too many truncation numbers.

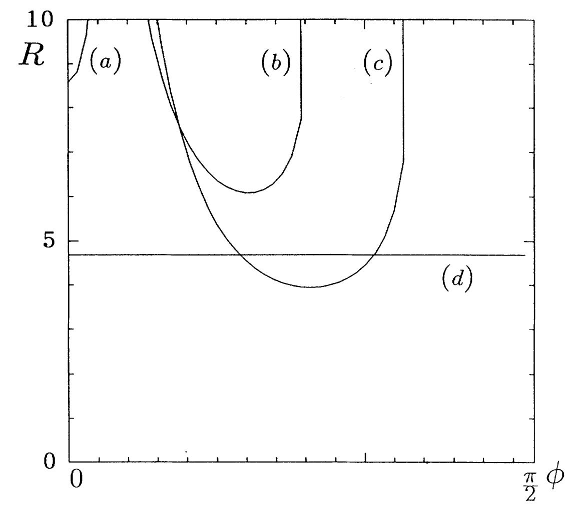

FIG. 3. The case of $k=0.6$. The Reynolds number for various θ. (a) $\theta=0.544$, (b) $\theta=0.801$, (c) $\theta=\pi/2$, which gives $R_c=3.93$, and (d) the periodic mode which is independent of θ and ϕ, and gives the 2D critical Reynolds number $R_{2c}=4.68$.

In order to confirm that this three-dimensional mode shows the negative eddy viscosity, we plot the dependence of the growth rate σ on the magnitude of the wave number ϵ in Fig. 4. Curves for various Reynolds numbers indicate the growth rate σ being in proportion to ϵ^2 in the vicinity of the origin and the proportional coefficient grows as the Reynolds number increases. The marginal Reynolds number is determined by the condition that this coefficient is zero. By using a log-log plot we obtain 2.0 for the value of this power.

In Fig. 2 the bold line with $\theta=\pi/2$, which is the direction of the shorter period of the main flow, indicates the minimum Reynolds number due to the three-dimensional disturbance. Here ϕ has been selected for each θ and k so as to give the minimum value of R. In other words, we draw the line: $R(k,\theta)=\min_\phi R(k,\theta,\phi)$. We find that the critical values are given by the three-dimensional disturbance in the range $0.54<k<0.71$. The minimum value

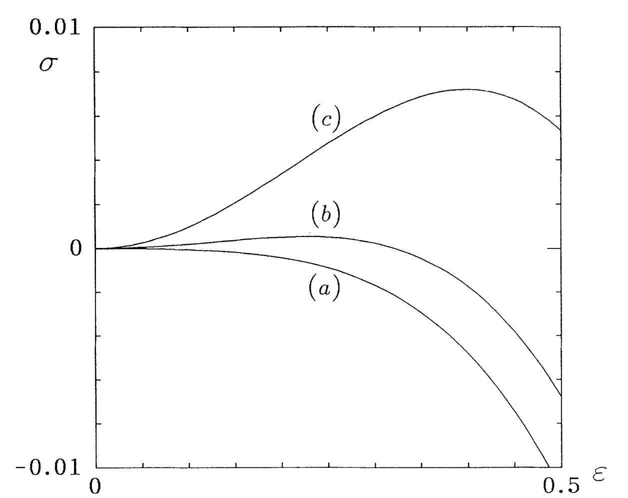

FIG. 4. $(\theta,\phi)=(\pi/2,0.815)$. (a) $R=3.90<R_c$, (b) $R=4.10>R_c$, and (c) $R=4.80>R_c$.

with $\theta=0$, which is the direction of the longer period of the main flow, is also represented by the bold line in Fig. 2. It is always larger than the value R_2 and both bold lines coincide at $k=1$ as we expect. From Table I we know that the relations $\theta_c=\pi/2$ and $\phi_c\approx\pi/4$ hold irrespective of k. This universal nature is greatly in contrast to the two-dimensional case. When the disturbances are limited to two-dimensional ones the critical large-scale modes depend on the anisotropic parameter k, i.e., $\theta_{2c}\approx\arctan(k)$ where θ_{2c} is the direction of the two-dimensional critical mode.

We have shown numerically that the three-dimensional negative eddy viscosity effect appears at the onset of the instability in the two-dimensional flow: $\Psi(x,y)=\cos kx\cos y$ in medium anisotropic range, $0.54<k<0.71$. It shows clearly that Squire's theorem cannot be extended to nonparallel two-dimensional flows even if we limit our concern to long-wave disturbances. The wave number of this mode is parallel to the y direction (i.e., the direction of the shorter period of the main flow) and the angle ϕ is close to $\pi/4$. The reason for the relation $\phi\approx\pi/4$ is as follows. When the magnitude of the two-dimensional exponents $\epsilon_2=\sqrt{\alpha^2+\beta^2}\ll\gamma$, the effect of γ is negligible, while γ works as damping when $\gamma\gg\epsilon_2$. The only possibility that the instability is induced by γ is the case $\gamma\approx\epsilon_2$. This result indicates that the *three*-dimensional large-scale structure may be organized in this direction spontaneously even in the small-scale two-dimensional forcing. The mere intrinsic properties of the two-dimensional flows can induce the three-dimensionality in the flow without additional three-dimensional external forcing. Therefore the two-dimensional flows which are uniform in the vertical direction by some mechanisms such as the condition of the Taylor-Proudman theorem [9] may break and create large-scale structures which are not parallel to the basic two-dimensional flows. It is of great interest to investigate theoretically and experimentally whether three-dimensional large-scale structures may generate in more general two-dimensional flows or not. Our results suggest that the ratio of the characteristic lengths of both the x and y directions in the flows is essential to its generation. We hope that some experiments in a rotating liquid with a long vertical uniformity which simulate geophysical phenomena [10] may be candidates for this purpose. In these experiments the two-dimensional secondary flow appears as a result of a primary instability of the parallel flow as usual. In some conditions a three-dimensional large-scale structure may be generated by the mechanism that we have presented here. Finally, we would like to stress that our results are entirely based on the numerical calculations of the eigenvalue problems. It is desirable to derive the same results by using some analytical methods. The method proposed by Dubrulle and Frisch [11] has been implemented in two- and three-dimensional flows [12,13]. Its application to our problem is left for a future work.

TABLE I. Values of (θ,ϕ) for the minimum Reynolds number to the three-dimensional disturbances.

k	θ	ϕ
0.30	$\pi/2$	0.818
0.40	$\pi/2$	0.817
0.50	$\pi/2$	0.817
0.55	$\pi/2$	0.816
0.60	$\pi/2$	0.815
0.65	$\pi/2$	0.814
0.70	$\pi/2$	0.813
0.80	$\pi/2$	0.809
0.90	$\pi/2$	0.803
1.00	$\pi/2$	0.798

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